Effective correlation times in turbulent scalar transport

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The effective correlation times in turbulent transport of passive scalar fields in the presence of a large-scale flow are investigated. For weak sweeping, the effective correlation times can be either enhanced or depleted depending on the detailed form of the autocorrelation function of turbulence. Strong large-scale sweeping always reduces the effective correlation times. This fact is exploited to derive explicit approximate formulas for the effective diffusivities. These expressions are then compared with numerical simulations of the Fokker-Planck equation for the passive scalar field. [S1063-651X(97)08711-4]

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INTRODUCTION

Turbulent transport of passive fields is of major importance in various domains ranging from astrophysics to geophysics. The main quantity of interest is typically the rate at which turbulence transports the scalar, e.g., a pollutant. For times large compared to those characteristic of the turbulent field, transport is diffusive and is characterized by effective diffusion coefficients (which are actually a second-order tensor). For high-Reynolds-number incompressible flow, the turbulent rates of transport typically greatly exceed the corresponding molecular rates. The molecular diffusivity D_0 is thus much smaller than the effective diffusivity and can be neglected. An exact formula obtained by Taylor [1] allows one to express the effective diffusivities as time integrals of Lagrangian correlations. The problem is that the relation between Lagrangian and Eulerian averages for generic turbulent flows is highly complicated. An exception is provided by the flows having short correlation times. For random flows δ-correlated in time, Lagrangian and Eulerian averages indeed coincide and the effective diffusivities can be determined exactly. The resulting expression depends on the energy spectrum only. For realistic flows, having finite correlation times, the effective diffusivities are generally dependent on all turbulence characteristics and no general approach for their calculations is known. Given the statistical properties of the turbulence, one would like to be able to calculate, at least approximately, the effective diffusivities. From the point of view of statistical field theory, this problem is equivalent to studying the infrared behavior of the mass operator appearing in the equation for the average Green's function. A fully consistent statistical theory, the *direct interaction approximation* (DIA), was developed in Ref. [2]. The idea essentially consists of neglecting any renormalization of the vertices. The DIA equation for the passive scalar problem was investigated in Ref. [3]. The quadratic equation for the average Green's function was solved numerically and the resulting numerical values of the effective diffusivities were found to be in fairly good agreement with the corresponding measured values.

Our aim here is to consider passive scalar transport in the presence of a mean flow. The advecting velocity field **v** is thus made by a constant (or slowly varying) part **U** and a fluctuating random part **u**, whose statistical properties are prescribed.

The above decomposition, which is standard, for instance, in the framework of mesoscale meteorology [4], implies that the small-scale eddies remain stationary while a slowly modification of the large-scale component occurs. In the realm of geophysics, this is a quite common feature, for example, inside the planetary boundary layer (PBL) [5], a thin atmospheric layer near the ground, where the airflow is strongly driven by sink- or source-forcing terms arising from the bottom boundary. In this atmospheric region, it then follows that dynamical effects induced by slowly variations of **U** on the small-scale velocity can be neglected as a reasonable approximation.

Inside the PBL, the response of turbulent flow to small-scale (dissipative) forcing (e.g., due to the orography) as well as, for example, radiative (driving) forcing arising from solar radiation reflected by the Earth must be accounted for. With respect to the former, the role of small-scale orography features consists [6] in extracting momentum from the atmosphere, primarily either through form drag, due to differential pressure across the object, or through gravity wave drag, by which internal gravity waves initiated by flow over the mountains propagate vertically and may extract momentum from the flow far aloft the surface forcing.

In general, such a type of surface forcing is not invariant under a Galilean transformation. As a consequence, such an invariance does not hold for the airflow inside the PBL. From this point of view, it may be interesting to investigate the effect of a slowly varying large-scale flow on the effective coefficients. In fact, for small-scale velocity field not invariant under Galilean transformations, the large-scale flow not only contributes (trivially) to a drift but it can act to modify the correlation times of the smaller-scale velocity field.

The first remark we shall exploit is that strong sweeping reduces the effective correlation times of the advecting turbulence **u**. Let us indeed denote by R_0 the correlation length of **u**. It is physically quite evident that for times larger than R_0/U the regions of **u** sampled by the scalar are essentially uncorrelated. This point can be analyzed in full detail by considering the simple case of parallel flows, as done in Sec. II A. On the other hand, from the results of Ref. [3], it is known that the DIA works better and better as the correlation time of the advecting flow reduces and becomes exact for flows δ-correlated in time. We combine these two remarks to derive explicit formulas for the effective diffusivities in the presence of a mean flow. These expressions are then compared with numerical simulations of the original passive scalar equation.

The paper is organized as follows. In Sec. I an extension of the theory of the passive scalar field outlined in Ref. [7] is presented in order to deal with velocity fields that vary also on large scales. The results obtained in that section are applied in Sec. II to the case of parallel flows. Different regimes corresponding to weak and strong sweeping are investigated concerning their effects on effective correlation times. The investigation is carried out both analytically and, in Sec. III, by numerical simulations performed on the Fokker-Planck equation for the passive scalar field. In Sec. IV the decorrelating effect associated with strong sweeping is exploited to derive explicit formulas for the effective diffusivities. Explicit approximate formulas are then evaluated by a comparison with numerical simulations of the Fokker-Planck equation for the passive scalar field. The final section is reserved for conclusions. In the Appendix the renormalized perturbation theory leading to the *direct interaction approximation* and to the *self-consistent method* for finding the effective diffusivities is presented.

I. THE MULTISCALE APPROACH IN THE PRESENCE OF LARGE-SCALE VELOCITY FIELDS

Multiscale techniques (see, e.g., Ref. [8]) have been used in Ref. [9] to show that the large-scale dynamics of the scalar, in the presence of scale separation with respect to the small-scale advecting velocity field, is governed by an effective equation that is always diffusive. The calculation of the effective diffusivity is reduced to the solution of one auxiliary equation.

In this section we generalize such results by considering velocity fields varying also on large scales. The goal is twofold: on the one hand to derive an effective equation governing the scalar field dynamics on large scales and, on the other hand, to obtain an equation for finding the effective diffusivity. As we shall see, the effective diffusivity depends not only on the turbulent velocity fields but also on the large-scale component $\mathbf{U}$. Such a dependence, and the related effects on the effective correlation times of turbulence, will be extensively investigated in the following sections.

The passive scalar field $\theta(\mathbf{x},t)$ obeys (see Ref. [10]) the Fokker-Planck equation

$$\partial_t \theta(\mathbf{x},t) + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\theta(\mathbf{x},t) = D_0 \Delta\, \theta(\mathbf{x},t). \tag{1}$$

The advecting velocity $\mathbf{v}(\mathbf{x},t)$ is incompressible and given by the sum of $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{U}(\mathbf{x},t)$. The first is periodic both in space (in a cell of size l) and in time (the technique can be extended with some modifications to handle the case of a random, homogeneous, and stationary velocity field). The second is the large-scale component of $\mathbf{v}$, which varies only on a typical scale L such that $l/L=\epsilon \ll 1$, where ϵ is the parameter controlling the scale separation. We are interested in the dynamics of the field $\theta(\mathbf{x},t)$ on large scales of $O(1/\epsilon)$. Simple physical reasoning suggests that the characteristic time scale associated with the diffusive dynamics should be of order $\epsilon^{-2} t_{\mathrm{eddy}}$, where $t_{\mathrm{eddy}} = l/u_0$ and u_0 is the characteristic amplitude of $\mathbf{u}$. Furthermore, the advection by $\mathbf{U}$ takes place on times $O(\epsilon^{-1})$.

In the spirit of multiscale methods (see Ref. [8]), in addition to the *fast* variables $\mathbf{x}$ and t, let us introduce *slow* variables as $\mathbf{X}=\epsilon\mathbf{x}$, $T=\epsilon^2 t$, and $\tau=\epsilon t$. The prescription of the technique is to treat the variables as independent. It follows that

$$\partial_i \mapsto \partial_i + \epsilon \nabla_i, \quad \partial_t \mapsto \partial_t + \epsilon\partial_\tau + \epsilon^2 \partial_T, \tag{2}$$

$$\mathbf{u} \mapsto \mathbf{u}(\mathbf{x},t), \quad \mathbf{U} \mapsto \mathbf{U}(\mathbf{X},T), \tag{3}$$

where ∂ and ∇ denote the derivatives with respect to fast and slow space variables, respectively. The solution is sought as a perturbative series

$$\theta(\mathbf{x},t;\mathbf{X},T;\tau) = \theta^{(0)} + \epsilon\theta^{(1)} + \epsilon^2\theta^{(2)} + \cdots, \tag{4}$$

where the functions $\theta^{(n)}$ depend *a priori* on both fast and slow variables. By inserting Eqs. (4) and (2) into Eq. (1) and equating terms having equal powers in ϵ, we obtain a hierarchy of equations. The solutions of interest to us are those having the same periodicities as the velocity field $\mathbf{u}(\mathbf{x},t)$.

It can be easily checked that the equations at order ϵ and ϵ^2 are

$$\begin{aligned}\partial_t \theta^{(1)} + (\mathbf{v}\cdot\boldsymbol{\partial})\,\theta^{(1)} - D_0\partial^2\theta^{(1)} = {}& -(\mathbf{v}\cdot\boldsymbol{\nabla})\,\theta^{(0)} \\ & -\partial_\tau\theta^{(0)} \quad \text{for } O(\epsilon),\end{aligned} \tag{5}$$

$$\begin{aligned}\partial_t \theta^{(2)} + (\mathbf{v}\cdot\boldsymbol{\partial})\,\theta^{(2)} - D_0\partial^2\theta^{(2)} = {}& -\partial_T\theta^{(0)} - (\mathbf{v}\cdot\boldsymbol{\nabla})\,\theta^{(1)} \\ & + D_0\nabla^2\theta^{(0)} \\ & + 2D_0(\boldsymbol{\partial}\cdot\boldsymbol{\nabla})\,\theta^{(1)} \\ & -\partial_\tau\theta^{(1)} \quad \text{for } O(\epsilon^2).\end{aligned} \tag{6}$$

Now we make use of the solvability conditions for Eqs. (5) and (6) (Fredholm alternative) and we exploit the fact that the solution $\theta^{(0)}$ goes to zero on a fast time scale [i.e., $\theta^{(0)}(\mathbf{x},t;\mathbf{X},T;\tau)=\theta^{(0)}(\mathbf{X},T;\tau)$; see [11] for details]. The linearity of Eq. (6) permits us to search for a solution in the form

$$\begin{aligned}\theta^{(1)}(\mathbf{x},t;\mathbf{X},T;\tau) = {}& \langle\theta^{(1)}\rangle(\mathbf{X},T;\tau) \\ & + \mathbf{w}(\mathbf{x},t;\mathbf{X},T)\cdot\boldsymbol{\nabla}\,\theta^{(0)}(\mathbf{X},T;\tau),\end{aligned} \tag{7}$$

where the angular brackets denote the average over the periodicities. The following equation is obtained:

$$\partial_T\theta^{(0)} + (\mathbf{U}\cdot\boldsymbol{\nabla})\langle\theta^{(1)}\rangle + \partial_\tau\langle\theta^{(1)}\rangle = \nabla_\alpha(D_{\alpha\beta}\nabla_\beta\theta^{(0)}), \tag{8}$$

where

$$D_{\alpha\beta}(\mathbf{X},T) = \delta_{\alpha\beta}D_0 - \langle u_\alpha w_\beta\rangle \tag{9}$$

is the eddy diffusivity (which is actually a second-order tensorial field) and $\mathbf{w}(x,t;\mathbf{X},T)$ has a vanishing average over the periodicities and satisfies the equation

$$\partial_t \mathbf{w} + [(\mathbf{u}+\mathbf{U})\cdot\boldsymbol{\partial}]\mathbf{w} - D_0 \partial^2 \mathbf{w} = -\mathbf{u}. \tag{10}$$

Note that when $\mathbf{U}$ is not a pure mean flow but depends on $\mathbf{X}$ and T Eq. (10) should be solved for every such value of $\mathbf{U}$. This point can be critical for computer memory costs when numerical methods are employed to solve such equation.

From Eq. (8) and from the solvability condition of Eq. (5)

$$\partial_\tau \langle \theta^{(0)} \rangle + (\mathbf{U}\cdot\boldsymbol{\nabla})\langle \theta^{(0)} \rangle = 0 \tag{11}$$

one obtains the Fokker-Planck equation for the field θ_L defined as $\theta_L \equiv \langle \theta^{(0)} \rangle + \epsilon \langle \theta^{(1)} \rangle$,

$$\partial_t \theta_L + (\mathbf{U}\cdot\boldsymbol{\partial})\theta_L = \partial_\alpha (D_{\alpha\beta} \partial_\beta \theta_L), \tag{12}$$

where the usual variables $\mathbf{x},t$ are used. From Eq. (12) it appears clearly that the energy $\int \theta_L^2 dV$ is controlled by the symmetric part of the eddy diffusivity. This can be easily shown to be positive, as in the case without large-scale streaming [9].

For such a purpose, let us consider the αth and the βth components of Eq. (10) and multiply by w_β and w_α, respectively. Taking the sum and averaging, the time derivative and the advective term vanish and we obtain

$$\frac{D_{\alpha\beta}+D_{\beta\alpha}}{2} = D_0 [\delta_{\alpha\beta} + \langle \boldsymbol{\partial} w_\alpha \cdot \boldsymbol{\partial} w_\beta \rangle]. \tag{13}$$

This tells us that the integral of θ_L^2 over the whole space is a decreasing function of time and the passive scalar cannot undergo amplification.

The calculation of eddy diffusivities is reduced to the solution of the auxiliary equation (10). Numerical methods are generally needed to solve it, but there are a few cases where one can obtain analytically the solution of Eq. (10). Among them, there are parallel flows at small scales in the presence of large-scale advecting velocity fields. Such a class of flows will be investigated in Sec. II.

In the case when $\mathbf{U}$ depends on space and time and it is not a pure streaming, there is a third range of characteristic scales, i.e., very large scales $\alpha \gg L$. Since Eq. (12) is a passive scalar equation, we expect that it should lead to a purely diffusive dynamics at those very large scales. The difference with respect to the usual passive scalar equation is that one has also to homogenize the diffusivity term, which also depends on space and time. Multiscale techniques can clearly be applied to derive the effective equations valid at very large scales $\alpha = L/\epsilon'$. By defining $\mathbf{X} = \epsilon' \mathbf{x}$ and $T = \epsilon'^2 t$ and using the same procedures previously discussed, we obtain a close evolution equation for the mean field $\theta_{\mathcal{L}} = \langle \theta_L^{(0)} \rangle$ (averages are performed over the cell of size L),

$$\partial_T \theta_{\mathcal{L}} = D^{(\mathcal{L})}_{\alpha\beta} \nabla_\alpha \nabla_\beta \theta_{\mathcal{L}}, \tag{14}$$

where the eddy-diffusivity tensor is given by

$$D^{(\mathcal{L})}_{\alpha\beta} = -\frac{\langle U_\alpha w_\beta \rangle + \langle U_\beta w_\alpha \rangle}{2} + \frac{\langle D_{\alpha r} \partial_r w_\beta \rangle + \langle D_{\beta s} \partial_s w_\alpha \rangle}{2} + \frac{\langle D_{\alpha\beta} \rangle + \langle D_{\beta\alpha} \rangle}{2}. \tag{15}$$

The vector field $\mathbf{w}$ has vanishing average over the periodicities and satisfies

$$\partial_t w_k + (\mathbf{U}\cdot\boldsymbol{\partial}) w_k - \partial_\alpha (D_{\alpha\beta} \partial_\beta w_k) = -U_k + \partial_i D_{ik}, \quad k=1,2,3. \tag{16}$$

In the cases when $D_{\alpha\beta}$ does not depend on space and time, Eqs. (15) and (16) reduce to the usual multiscale equations for the passive scalar (see, e.g., Ref. [7]). In the next section we shall show a simple flow that permits us to calculate analytically the solution of Eq. (16).

II. WEAK AND STRONG SWEEPING: TWO DIFFERENT EFFECTS ON THE CORRELATION TIMES

We apply the results outlined in the preceding section to the simple case of random parallel flows [12] in the presence of a large-scale velocity field. Such idealized flows permit us to obtain analytical expressions for the correlation time of turbulence and thus to clearly capture the physical mechanisms associated with the large-scale streaming.

A. Effective correlation times for parallel flows

In three dimensions, random parallel flows, in the presence of a large-scale advecting velocity field $\mathbf{U}(\mathbf{X},T)$, are defined as

$$\mathbf{v}(\mathbf{x},t;\mathbf{X},T) = \mathbf{u}(\mathbf{x},t) + \mathbf{U}(\mathbf{X},T), \tag{17}$$

with

$$\mathbf{u}(\mathbf{x},t) = (u(y,z,t),0,0), \quad \mathbf{U}(\mathbf{X},T) = (0,U(X,Z,T),0). \tag{18}$$

Here $u(\mathbf{x},t)$ is random, homogeneous, and stationary and u and U do not depend on x and Y, respectively, on account of incompressibility.

The solution of the auxiliary equation (10) is obtained by noting that such an equation can be reduced to a form involving only the w_1 component. The latter (which is in the fast variables) can be easily solved in Fourier space. The solution is

$$\hat{w}_1(\mathbf{k},\omega;\mathbf{X},T) = \frac{-\hat{u}(\mathbf{k},\omega)}{i\omega + k^2 D_0 + i\mathbf{U}\cdot\mathbf{k}}, \tag{19}$$

which, after introducing the advective-diffusion propagator

$$\hat{G}(\mathbf{k},\omega;\mathbf{X},T) = \frac{1}{i\omega + k^2 D_0 + i\mathbf{U}\cdot\mathbf{k}},$$

takes the form

$$\hat{w}_1(\mathbf{k},\omega;\mathbf{X},T) = -\hat{u}(\mathbf{k},\omega)\hat{G}(\mathbf{k},\omega;\mathbf{X},T). \tag{20}$$

From Eq. (9) the eddy diffusivity is easily found

$$D_\perp(\mathbf{X},T)=D_0+\int\int_{-\infty}^{+\infty}\hat{E}(\mathbf{k},\omega)\hat{G}(\mathbf{k},\omega;\mathbf{X},T)d\mathbf{k}\,d\omega,$$

$$D_\parallel=D_0. \tag{21}$$

Hereafter, $D_\perp$ and $D_\parallel$ are the components of the eddy-diffusivity tensor orthogonal and parallel to the direction of the velocity component $\mathbf{U}$. Furthermore, we have defined $\hat{E}(\mathbf{k},\omega)=\langle|\hat{u}(\mathbf{k},\omega)|^2\rangle$.

The expression for $\hat{G}(\mathbf{k},\omega;\mathbf{X},T)$ given by Eq. (20) can be recast in the more convenient form

$$\hat{G}(\mathbf{k},\omega;\mathbf{X},T)=\int_0^{+\infty}e^{-[i\omega+k^2D_0+i\mathbf{U}\cdot\mathbf{k}]\alpha}d\alpha. \tag{22}$$

By inserting Eq. (22) into Eq. (21) and assuming separability in $\hat{E}(\mathbf{k},\omega)$ [i.e., $\hat{E}(\mathbf{k},\omega)=\hat{E}(\mathbf{k})\hat{S}(\omega)$] the following expression for $D_\perp$ is found:

$$D_\perp(\mathbf{X},T)=D_0+\int_0^\infty S(t)\int\hat{E}(\mathbf{k})e^{-[k^2D_0+i\mathbf{U}\cdot\mathbf{k}]t}dt\,d\mathbf{k}, \tag{23}$$

where $\hat{E}(\mathbf{k})$ is the energy spectrum and $S(t)$ is the temporal part of the autocorrelation function.

Note that for strong sweeping the above relation holds for general flows. This point can be easily checked by observing that after imposing $|\mathbf{U}|\gg|u|$ in Eq. (10) one obtains the expression (20) for $\hat{w}_1$.

Despite the simplicity of the flow here considered, expression (23) permits us to analyze the two different regimes associated with weak and strong sweeping, respectively. To start our analysis, let us consider for simplicity the case when D_0 vanishes, to obtain

$$D_\perp(\mathbf{X},T)=\int_0^\infty S(t)E(Ut)dt, \tag{24}$$

where $E(Ut)$, the inverse Fourier transform of $\hat{E}(\mathbf{k})$, is given by the two-point correlation function

$$E(Ut)=\langle u(\mathbf{x}+\mathbf{U}t)u(\mathbf{x})\rangle, \tag{25}$$

which does not depend on $\mathbf{x}$ due to homogeneity. When $t=0$, $E(Ut)\equiv E_0$ is the energy of turbulence.

Thanks to the properties of correlation functions, when $t\geqslant 0$ we can write down

$$E(Ut)=E_0C_U(t)\quad\text{with }|C_U(t)|\leqslant 1. \tag{26}$$

By inserting Eq. (26) into Eq. (24) we obtain

$$D_\perp(U)=E_0\int_0^\infty S_{\text{eff}}(t)dt\equiv E_0\tau_{\text{eff}}, \tag{27}$$

where $S_{\text{eff}}(t)=S(t)C_U(t)$ and the effective correlation time is defined as

$$\tau_{\text{eff}}=\int_0^\infty S_{\text{eff}}(t)dt, \tag{28}$$

while the correlation time of the flow is defined as $\tau_0=\int_0^\infty S(t)dt$. Sweeping will therefore decrease (increase) the correlation time when $\tau_{\text{eff}}<\tau_0$ ($\tau_{\text{eff}}>\tau_0$).

It immediately follows from expression (27) that if $C_U(t)$ does not have anticorrelated regions [i.e., $C_U(t)\geqslant 0\ \forall t$], then

$$\tau_{\text{eff}}<\tau_0 \tag{29}$$

as a consequence of the second relation in Eq. (26). We remark that when $C_U(t)\geqslant 0$, relation (29) is fulfilled independently of the sweeping intensity.

From Eq. (23) it is easy to verify that for strong sweeping (i.e., $U\gg R_0/\tau_0$) one obtains $D_\perp\propto 1/U^2$. For the temporal part of the autocorrelation functions taken as, in Ref. [13],

$$S(t)=e^{-|t|/\tau_0}, \tag{30}$$

the expression for the effective correlation time reads

$$\tau_{\text{eff}}=\frac{\overline{1/k^2}}{\tau_0}\left(\frac{1}{U}\right)^2, \tag{31}$$

where $\overline{1/k^2}=(1/u_0^2)\int(1/k^2)\hat{E}(k)dk$. The effective correlation time τ_{eff} thus tends to zero when $U\to\infty$, independently of the form of $C_U(t)$ and $S(t)$. Such a result holds indeed for general flows.

Enhancement of correlation times can only takes place when the sweeping is weak and anticorrelated regions are present. The importance of anticorrelated regions already has been pointed out in Ref. [14].

To concentrate our attention on weak sweeping, we can treat perturbatively the $\mathbf{U}$ term in the exponential on the right-hand side of Eq. (23) and consider second-order times. Expression (23) reduces then to

$$D_\perp(\mathbf{X},T)=E_0\tau_0-\frac{U^2}{2}\int_0^\infty k^2\hat{E}(k)dk\int_0^\infty S(t)t^2dt. \tag{32}$$

The integral $\int_0^\infty S(t)t^2dt$ can be negative for admissible correlation functions [15]. As an example we can choose

$$S(t)=e^{-|t|/T}\cos(\Omega t). \tag{33}$$

Here the correlation time of the flow $\tau_0\equiv\int_0^\infty S(t)dt=T/(1+\Omega^2T^2)$ and

$$\int_0^\infty S(t)t^2dt=-\frac{2T^3}{(1+\Omega^2T^2)^3}(3\Omega^2T^2-1)=-6\tau_0^3\left[\frac{T}{\tau_0}-\frac{4}{3}\right]. \tag{34}$$

Expression (34) turns out to be negative when $T>\frac{4}{3}\tau_0$. Finally, by inserting Eq. (34) into Eq. (32), the expression for $D_\perp$ becomes

$$D_\perp=E_0\tau_{\text{eff}}$$

with

$$\tau_{\text{eff}}=\tau_0+3\tau_0^3\left(\frac{T}{\tau_0}-\frac{4}{3}\right)\frac{U^2}{E_0}\int k^2\hat{E}(k)dk. \tag{35}$$

We conclude that

$$\tau_{\rm eff} > \tau_0 \quad \text{if} \quad T > \tfrac{4}{3}\tau_0 .$$

Thus an enhanced correlation time can occur in the presence of weak sweeping.

The expressions for $\tau_{\rm eff}$ obtained in this section have been derived by ''observing'' the dynamics of the passive scalar field on scale of the order of L, the length scale of the velocity field $\mathbf{U}$. For such a range of scales, eddy diffusivities and effective correlation times are smoothly dependent on spatial and temporal variables (i.e., on $\mathbf{X}$ and T).

Our aim now is to investigate effective correlation times by observing the dynamics on scales much larger than L. As already shown in Sec. I, the dependence on variables $\mathbf{X}$ and T is averaged out.

In order to tackle the problem analytically, we focus our attention on the flow (18) with $\mathbf{U}$ depending only on Y. Equation (12) then becomes

$$\partial_t \theta_L = \partial_x (D_\perp \partial_x \theta_L), \tag{36}$$

with $D_\perp$ given by Eq. (24).

For time-independent solutions, from Eq. (16) it follows that

$$\partial_x (D_\perp \partial_x w) = -\partial_x D_\perp , \tag{37}$$

where w stays for w_1. This equation can be integrated to obtain

$$w(x) = \int_0^x \frac{\alpha}{D_\perp(y)}\, dy + \beta - x, \tag{38}$$

where constants α and β can be calculated by imposing the periodicity of w [i.e., $w(x)=w(x+L)$] and $\int_0^L w(y)dy = 0$ (i.e., $\langle w\rangle = 0$). From both conditions we obtain

$$\alpha = \frac{1}{\langle D_\perp \rangle}, \quad \beta = \frac{L}{2} - \frac{1}{L\langle 1/D_\perp\rangle} \int_0^L \int_0^x \frac{1}{D_\perp(y)}\, dx\, dy.$$

As one can easily check, by plugging Eq. (38) into Eq. (15) the eddy diffusivity is found:

$$D^{(\mathcal{L})} = \frac{1}{\langle 1/D_\perp \rangle}. \tag{39}$$

For strong sweeping [i.e., $D_\perp(U) = (u_0^2 \overline{1/k^2}/\tau_0 U^2]$ we obtain

$$D^{(\mathcal{L})} = u_0^2 \frac{\overline{1/k^2}}{\tau_0} \frac{1}{\langle U^2 \rangle} \equiv u_0^2 \tau_{\rm eff}^{(\mathcal{L})} \quad \text{with} \quad \tau_{\rm eff}^{(\mathcal{L})} = \frac{\overline{1/k^2}}{\tau_0} \frac{1}{\langle U^2\rangle}. \tag{40}$$

Thus fluctuations in the large-scale velocity field reduce the effective correlation time. Note that the decorrelating effect is present also if $\langle \mathbf{U} \rangle = 0$.

In the regime characterized by weak sweeping, $D_\perp$ is given by

$$D_\perp(U) = E_0 \tau_{\rm eff}(U)$$

with

$$\tau_{\rm eff}(U) = \tau_0 - \frac{U^2}{2u_0^2} \int_0^\infty k^2 \hat{E}(k) dk \int_0^\infty S(t) t^2 dt \tag{41}$$

[see Eq. (32)]. From Eq. (39) the eddy diffusivity is found:

$$D^{(\mathcal{L})} = \frac{u_0^2}{\left\langle \dfrac{1}{\tau_{\rm eff}(U)} \right\rangle} \equiv u_0^2 \tau_{\rm eff}^{(\mathcal{L})} \quad \text{with} \quad \tau_{\rm eff}^{(\mathcal{L})} = \frac{1}{\left\langle \dfrac{1}{\tau_{\rm eff}(U)} \right\rangle}. \tag{42}$$

We conclude that $\tau_{\rm eff}^{(\mathcal{L})} > \tau_0$ when $\int_0^\infty S(t) t^2 dt < 0$. As a consequence, the mechanism that can work to enhance the correlation time for weak sweeping does not depend on the scale on which one observes the dynamics of the passive scalar field.

III. DEPLETION OF CORRELATION TIMES FOR MORE GENERAL FLOWS: NUMERICAL INVESTIGATION

The previous arguments applied to the simple case of random parallel flow have revealed the mechanisms that can act either to enhance or to deplete the effective correlation times of turbulence. In particular, for general flows we have shown that for strong sweeping depletion always occurs, independently of the form of the autocorrelation function of turbulence. Here we are interested in analyzing the reduction of the correlation time for more general flows and moderate sweeping amplitudes. Since such an investigation is not accessible analytically [in this case in fact the nonlinear term $\mathbf{u}\cdot\boldsymbol{\partial}\mathbf{w}$ cannot be neglected in Eq. (10)], we have decided to perform direct numerical simulations of the original Fokker-Planck equation (1).

Integration (without space symmetries) of the stochastic partial differential equation (1) is carried out in two dimensions on a square domain with side 2π. Given the spatial periodic boundary conditions, we can solve the equation by a pseudospectral method (see Ref. [16]). Dealiasing is obtained by a proper circular truncation, which ensures better isotropy of numerical treatment.

Time marching is performed using a leapfrog scheme mixed with a predictor-corrector scheme (see Ref. [17]) at regular intervals. In all the cases to be reported here and in the subsequent sections, we have worked with a resolution of 512×512, which is found to be always adequate working with (adimensional) molecular diffusivity $\widetilde{D}_0 \equiv D_0/2\pi u_0 = 2\times10^{-3}$. The system evolution is computed for $60\tau_0$ with a time step $\Delta t = \tau_0/100$, depending on τ_0, the correlation time of the turbulent fluctuations.

The advecting velocity field is given by the sum of a constant part $\mathbf{U}$ and a zero mean Gaussian random field $\mathbf{u}$, statistically homogeneous, stationary, and homogeneous with an asymptotic spectrum of the Kraichnan-Batchelor type

$$\hat{E}(k) = 2p_0^2 u_0^2 k^{-3} \quad \text{for } k \geqslant p_0$$
$$\hat{E}(k) = 0 \quad \text{for } k < p_0 . \tag{43}$$

The velocity $\mathbf{U}$ (held constant during the marching of each simulation) is posed along the x axis.

The time dependence of the two-point velocity correlators is exponential: $S(t) = e^{-|t|/\tau_0}$. To obtain such a temporal behavior a digital filter transfer function has been realized (see Ref. [18] for details). More precisely, for each time step the stream function of the velocity field $\mathbf{u}$ has been defined in Fourier space as

$$\hat{\psi}(\mathbf{k},t+\Delta t)=a\,\hat{\psi}(\mathbf{k},t)+\sqrt{1-a^2}\,\hat{\eta}(\mathbf{k},t), \tag{44}$$

where $a=e^{-\Delta t/\tau_0}$ and the $\hat{\eta}(\mathbf{k},t)$'s are zero-mean complex Gaussian random variables chosen independently for each $\mathbf{k}$ [except for the Hermitian symmetry $\hat{\eta}(-\mathbf{k},t)=\hat{\eta}^*(\mathbf{k},t)$] and at each time step. Their variance is

$$\langle|\hat{\eta}(\mathbf{k},t)|^2\rangle=\frac{\hat{E}(k)}{k^3}. \tag{45}$$

It is easy to verify from Eqs. (44) and (45) that the following relation holds:

$$\langle\hat{\psi}(\mathbf{k},t)\hat{\psi}(\mathbf{k}',t')\rangle=\delta(\mathbf{k}+\mathbf{k}')\,\frac{\hat{E}(k)}{k^3}\,e^{-|t-t'|/\tau_0}. \tag{46}$$

The initial condition of the passive scalar field is chosen

$$\theta(\mathbf{x},0)=4+\cos(\boldsymbol{\alpha}\cdot\mathbf{x})+\cos(\boldsymbol{\beta}\cdot\mathbf{x}), \tag{47}$$

with the wave numbers $\boldsymbol{\alpha}=(2,0)$ and $\boldsymbol{\beta}=(0,2)$. Initial data are thus concentrated only on the large scales $\alpha\ll p_0$ and $\beta\ll p_0$.

We know from the previous multiscale theory that the temporal evolution of θ in the infrared limit ($t\gg\tau_0$ and $k\ll p_0$) is purely diffusive. It follows that

$$|\hat{\theta}(\mathbf{k},t)|^2\propto e^{-2D_{\alpha\beta}k_\alpha k_\beta t} \tag{48}$$

and in particular

$$|\hat{\theta}(\boldsymbol{\alpha},t)|^2\propto e^{-2D_\parallel\alpha^2 t},\quad |\hat{\theta}(\boldsymbol{\beta},t)|^2\propto e^{-2D_\perp\beta^2 t}. \tag{49}$$

The parallel and transverse effective diffusivities $D_\parallel$ and $D_\perp$ can thus be easily measured by performing a log-linear fit of $|\hat{\theta}(\boldsymbol{\alpha},t)|^2$ and $|\hat{\theta}(\boldsymbol{\beta},t)|^2$ vs t. The temporal range of the fit should be chosen at times large enough for the diffusive behavior (49) to take place.

In order to investigate the effect of U on the effective correlation time of turbulent fluctuations, we have measured $D_\parallel$ and $D_\perp$ for simulations with different values of U and S. The latter is the Strouhal number defined as $S=u_0\tau_0p_0/2\pi\equiv\tau_0/t_0$, where $t_0=2\pi/u_0p_0$ is the turnover time of the flow.

In Fig. 1 we show the eddy-diffusivity map as a function of U/u_0 and S. Two remarks are in order. First, we note [principally from Fig. 1(a)] that points belonging to the regions of the S-U/u_0 plane with large S and nonzero U are equivalent (in the sense that they have the same eddy diffusivity) to points with $U'=0$ and S' always smaller than S, namely,

$$D(U,S)=D(0,S_{\text{eff}})\quad\text{with}\quad S_{\text{eff}}<S, \tag{50}$$

which is a clear signature of the decorrelating effect due to the sweeping. Since u_0 and p_0 are kept fixed, the previous inequality is indeed equivalent to $\tau_{\text{eff}}<\tau_0$.

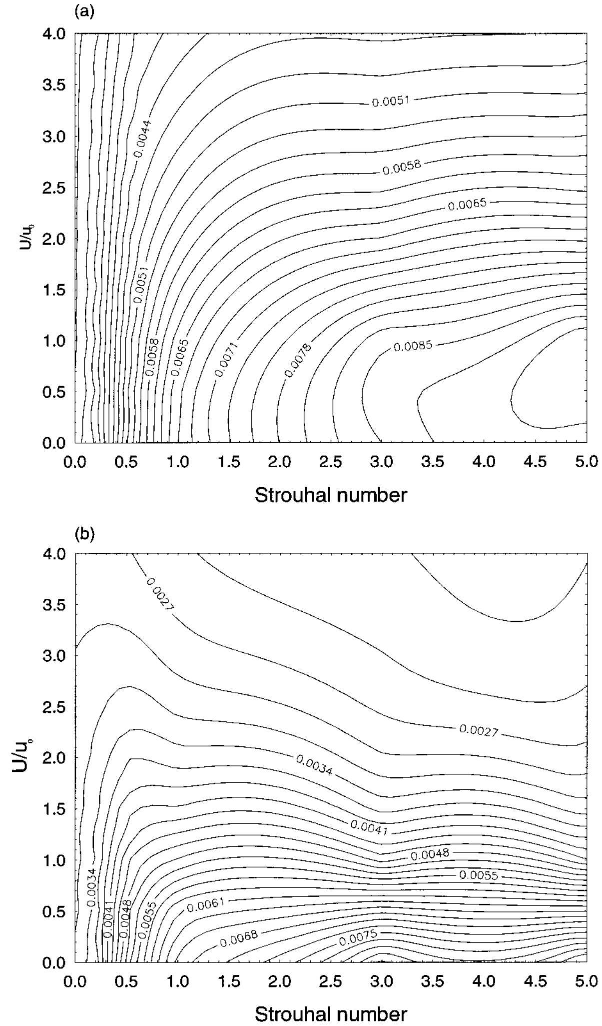

FIG. 1. Contour map for the diagonal (adimensional) components of the turbulent diffusivity as a function of the ratio U/u_0 and of the Strouhal number. (a) $D_\parallel/2\pi u_0$ and (b) $D_\perp/2\pi u_0$. The contour interval is 0.23×10^{-3}.

The second remark is that when $U/u_0\gtrsim2$ even points with $S=5$ are mapped to regions with $S_{\text{eff}}<1$. It follows that either moderate or strong sweeping makes the calculation of the diffusivities essentially equivalent to the analysis of a flow with small S. This is the realm of application of perturbative techniques. This remark will be exploited in the next section to derive explicit formulas for the effective diffusivities in the presence of moderate or strong mean flows.

IV. EXPLICIT EXPRESSIONS FOR EDDY DIFFUSIVITIES

As discussed in more detail in the Appendix, when the correlation time of turbulence is small, the calculation of eddy diffusivities can be tackled self-consistently. In particular, the following integral equation is derived:

$$D_{\alpha\beta}=\delta_{\alpha\beta}D_0+\frac{1}{(2\pi)^d}\int d\mathbf{q}\,\frac{(q^2\delta_{\alpha\beta}-q_\alpha q_\beta)f(q)}{1/\tau_0+i\mathbf{q}\cdot\mathbf{U}+D_{rs}q_rq_s}, \tag{51}$$

where the autocorrelation function $S(t)$ is given by Eq. (30). If the mean velocity field component is zero and if $D_{\alpha\beta}$ is isotropic, $D_{\alpha\beta}=D\delta_{\alpha\beta}$, Eq. (51) reduces to

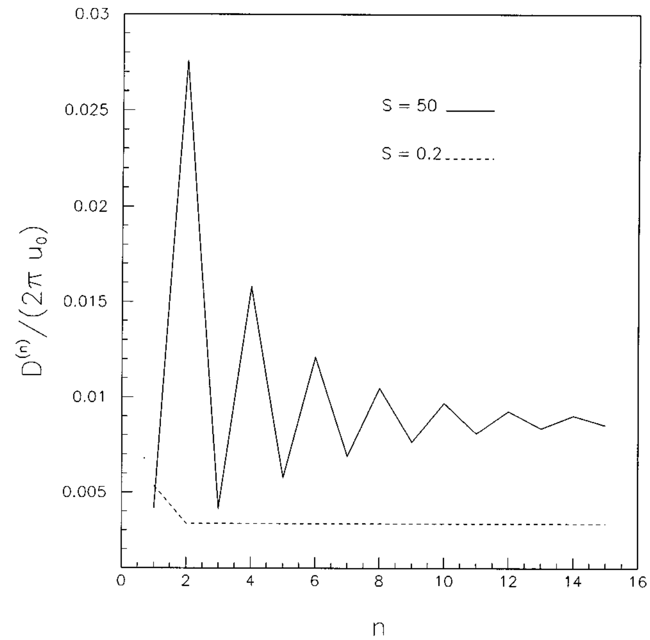

FIG. 2. Convergence profile $D^{(n)}/2\pi u_0$ of the (adimensional) turbulent diffusivity for $S=0.2$ (dashed curve) and $S=50$ (solid curve). Iterations have been performed on Eq. (52) in the two-dimensional case with the spectrum (43). The first guess field is $D^{(0)}=D_0+\frac{1}{2}u_0^2\tau_0$.

$$D=D_0+\frac{1}{d}\int_0^\infty dq\,\frac{\hat{E}(q)}{1/\tau_0+Dq^2}, \tag{52}$$

where

$$\hat{E}(q)=\frac{f(q)q^{d+1}}{[(4-d)\pi^{2d-3}]},\quad d=2,3 \tag{53}$$

is the energy spectrum. Since $\hat{E}(q)\geqslant 0$, the method of *chain fractions* is applicable to Eq. (52) for finding D. Given a first guess $D^{(0)}$ for D, the next approximation

$$D^{(1)}=\delta_{\alpha\beta}D_0+\frac{1}{d}\int_0^\infty dq\,\frac{\hat{E}(q)}{1/\tau_0+D^{(0)}q^2} \tag{54}$$

is smaller than the true value D. The subsequent iterations result in a convergent series of approximations for D. The true D lies, in all cases, between the values of two subsequent iterations. We remark that when $\tau_0\ll 1$ (i.e., $S\to 0$), relation (52) gives immediately

$$D=D_0+\frac{1}{d}\,u_0^2\tau_0, \tag{55}$$

the well-known result corresponding to turbulence with a short memory (i.e., δ-correlated in time).

When $S\lesssim 1$, successive iterations of Eq. (52) starting from the first guess (55) result in a rapidly convergent series of approximations for the true value D. As an example, in Fig. 2 the convergence profile $D^{(n)}$ is shown as obtained by performing successive iterations of Eq. (52) in the two-dimensional case with $S=0.2$ (dashed curve). As we can see, two iterations are sufficient for essentially perfect agreement.

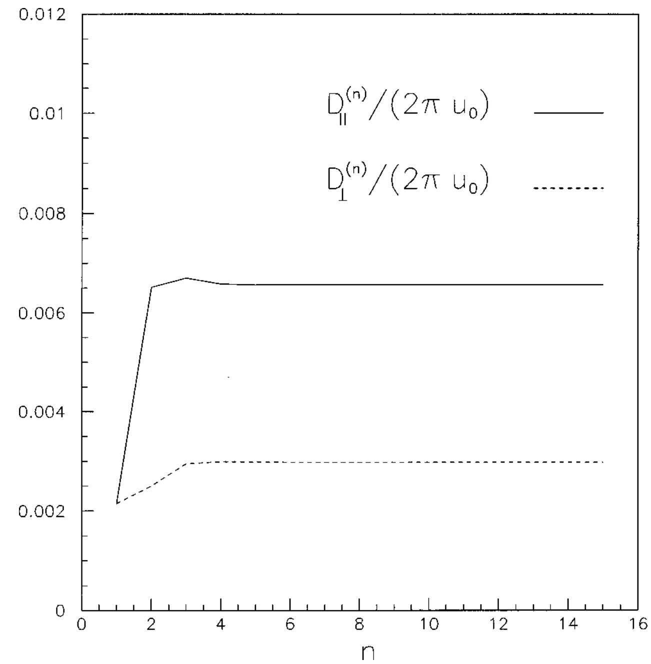

FIG. 3. Convergence profile for the diagonal (adimensional) component $D_\parallel^{(n)}/2\pi u_0$ (solid curve) and $D_\perp^{(n)}/2\pi u_0$ (dashed curve) of the turbulent diffusivity. Iterations have been performed on Eq. (51) in the two-dimensional case with the spectrum (43), $U/u_0=2.0$, and $S=50$. First guess is given by $D_\parallel^{(0)}=D_\perp^{(0)}=D_0+\frac{1}{2}u_0^2\tau_0$.

For turbulent spectra with large values of the parameter S, convergence of successive iterations is achieved more slowly, as should be expected. This aspect is detectable by observing Fig. 2 (solid curve) relative to the convergence profile $D^{(n)}$ for $S=50$.

The number of iterations to achieve convergence is thus directly proportional to S. The remark we shall exploit is that a similar situation occurs in the presence of strong sweeping, but where S is replaced by S_{eff}, as stressed by observing Fig. 3, in which the convergence profiles $D_\parallel^{(n)}$ (solid curve) and $D_\perp^{(n)}$ (dashed curve) for $U/u_0=2.0$ are shown. Iterations have been performed in the two-dimensional case starting from the first guess $D_\perp^{(0)}=D_\parallel^{(0)}=D_0+\frac{1}{2}u_0^2\tau_0$ on the self-consistent equation (51) with energy spectrum given by Eq. (43). The Strouhal number is $S=50$.

The convergence profiles can be compared with that already shown in Fig. 2 relative to the same S but with $U=0$. As one can see, convergence for $U/u_0=2$ is achieved much faster than in the case with $U=0$. The same conclusion holds in general when the sweeping is strong enough.

Now we can exploit the above remarks concerning the effect of strong sweeping on the convergence profiles of turbulent diffusivities in order to obtain their approximate explicit expressions valid for all S's. For such a purpose, let us insert into Eq. (51) the first guess $D_\parallel^{(0)}=D_\perp^{(0)}=D_0+\underline{(1/d)}u_0^2\tau_{\mathrm{eff}}^\infty$, with $\tau_{\mathrm{eff}}^\infty$ suggested by Eq. (31): $\tau_{\mathrm{eff}}^\infty=\overline{(1/k^2)}/\tau_0U^2$. Due to the isotropy of the first guess here considered, the angular integration in Eq. (51) is easily performed. More accurate expressions for the turbulent diffusivities are thus found for the two-dimensional case

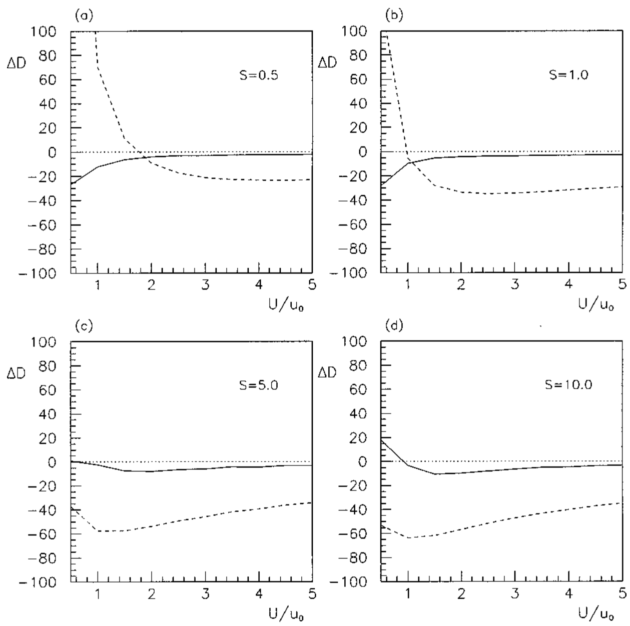

FIG. 4. Behavior of the percentage error ΔD (see the text) versus the ratio U/u_0, for different values of the Strouhal number: (a) $S=0.5$, (b) $S=1.0$, (c) $S=5.0$, and (d) $S=10.0$. Solid curves correspond to ΔD obtained with $D_\parallel$ given by Eq. (56). Dashed curves are relative to the first guess.

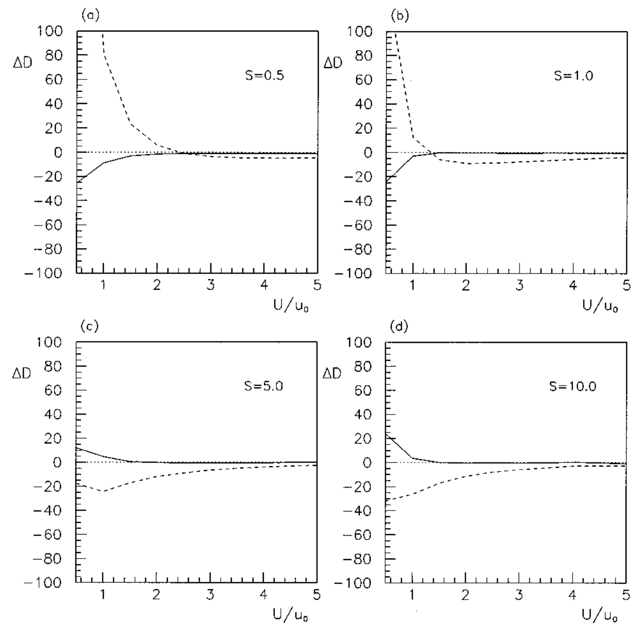

FIG. 5. Same as in Fig. 4, but $D_\parallel$ is replaced by $D_\perp$ in the definition of ΔD.

$$D_\parallel = D_0 + \int_0^\infty dp\, \frac{\hat{E}(p)}{p^2 U^2}\left(\sqrt{f^2+p^2U^2}-f\right), \tag{56}$$

$$D_\perp = D_0 + \int_0^\infty dp\, \frac{\hat{E}(p)}{p^2 U^2}\left(f - \frac{f^2}{\sqrt{f^2+p^2U^2}}\right) \tag{57}$$

and for the three-dimensional case

$$D_\parallel = D_0 + \frac{1}{4}\int_0^\infty dp\, \frac{\hat{E}(p)}{pU}\left\{2\left[1+\left(\frac{f}{pU}\right)^2\right]\arctan\left(\frac{pU}{f}\right) - 2\,\frac{f}{pU}\right\}, \tag{58}$$

$$D_\perp = D_0 + \frac{1}{4}\int_0^\infty dp\, \frac{\hat{E}(p)}{pU}\left\{\left[1-\left(\frac{f}{pU}\right)^2\right]\arctan\left(\frac{pU}{f}\right) + \frac{f}{pU}\right\}, \tag{59}$$

where

$$f = \frac{1}{\tau_0} + \left(D_0 + \frac{1}{d}\,u_0^2 \tau_{\mathrm{eff}}^\infty\right)p^2 \quad \text{with} \quad \tau_{\mathrm{eff}}^\infty = \frac{\overline{1/k^2}}{\tau_0}\left(\frac{1}{U}\right)^2. \tag{60}$$

The above expressions for the turbulent diffusivities in the two-dimensional case have been compared with measured values $D_\parallel^{\mathrm{FP}}$ and $D_\perp^{\mathrm{FP}}$ obtained by performing direct numerical simulations (see Sec. III) of the original Fokker-Planck equation (1).

Figure 4 shows the behavior of the error ΔD (percentage) defined as $\Delta D = (D_\parallel - D_\parallel^{\mathrm{FP}})/D_\parallel^{\mathrm{FP}}$ versus the ratio U/u_0 for different values of the Strouhal number S: (a) $S=0.5$, (b) $S=1.0$, (c) $S=5.0$, and (d) $S=10.0$. Solid curves correspond to ΔD obtained with $D_\parallel$ given by Eq. (56). Dashed curves are relative to ΔD calculated with the first guess $D_\parallel^{(0)}$. Figure 5 is the same as Fig. 4, but it is relative to the transverse component $D_\perp$.

From such figures several observations are worth mentioning. In all cases reported in the figures it is evident that approximate explicit expressions work better and better as the ratio U/u_0 increases. For $U/u_0 \gtrsim 1$ errors between turbulent diffusivity values predicted by analytic expressions and those obtained by direct numerical simulations of the Fokker-Planck equation are always smaller than 10%. Errors become smaller than 5% for $U/u_0 \gtrsim 3$. The quality of the approximation reduces when $U/u_0 \lesssim 1$, in agreement with the asymptotic character of the first guess employed (i.e., the first guess corresponding to strong sweeping regimes).

Furthermore, we notice that the agreement between values from our formulas and the measured values is better for the transverse components $D_\perp$ than for the parallel components $D_\parallel$. This is not surprising: The behavior of $D_\parallel$ for large $\mathbf{U}$ goes to zero as $1/U$, while the first guess for $D_\parallel$ employed here goes to zero as $1/U^2$, which is the behavior of $D_\perp$ for strong sweeping.

Our explicit expressions for effective diffusivities can be generalized to the case when $\mathbf{U}$ is not a pure streaming but varies on large scales [in the multiscale formalism $\mathbf{U}(\mathbf{X},T)$]. Equation (40) suggests that U can be replaced by $\sqrt{\langle U^2\rangle}$ in explicit formulas. By performing such a substitution, effective diffusivities given by explicit expressions must be seen as relative to the dynamics of the scalar field on a scale much larger than those on which $\mathbf{U}$ varies.

CONCLUSION

The effective correlation times in turbulent transport of a passive scalar field in the presence of a large-scale flow have been investigated. Approximate explicit expressions for the effective diffusivities have been proposed. In the present paper, the advecting velocity field is made by a constant (or varying on large scales) part and a fluctuating random part, with given statistical properties. By using multiscale methods, we have performed a generalization of the theory in Ref. [9] (see also Ref. [7]) showing that, in the presence of a large-scale velocity field and with scale separation, the large-scale scalar field dynamics is governed by an effective Fokker-Planck equation characterized by an effective turbulent diffusivity, which is actually a second-order tensor field (smoothly dependent on space and temporal coordinate). The calculation of the latter is reduced to the solution of one auxiliary equation. The effective diffusivity is always larger than the molecular one, i.e., incompressible flow enhanced diffusion, as expected.

The dependence of the effective diffusivities on the large-scale velocity field has been investigated both analytically and by numerical simulations performed on the original Fokker-Planck equation governing the scalar dynamics at full scales. With respect to the analytical analysis, the results of the theory carried out via the multiple scale method have been applied to the simple case of parallel flow in the presence of a large-scale advecting velocity field. In such a case, we have obtained the solution of the auxiliary equation analytically and exact expressions for the effective correlation time of turbulence have been found. The effect of the large-scale velocity field has been investigated. We have focused our attention on the cases of strong and weak sweeping, respectively. For strong sweeping decorrelation always occurs independently of the form of the autocorrelation function of turbulence: The effective correlation time tends to zero when the sweeping becomes strong. The same result holds for general flows. For weak sweeping, the correlation time can be either enhanced or depleted.

In order to investigate the reduction of the correlation time for more general flows and moderate sweeping amplitudes, we have performed direct numerical simulations of the original Fokker-Planck equation governing the scalar dynamics at full scales. Results have confirmed those obtained with the theoretical analysis performed on parallel flows.

The decorrelating effect associated with the strong sweeping has been exploited to derive explicit approximate formulas for the effective diffusivities. The basic idea consists in observing that the *direct interaction approximation* for the scalar problem works better and better as the correlation time of the advecting flow reduces and it becomes exact for flows δ-correlated in time. This allowed us to derive explicit formulas for the effective diffusivities, which were then compared with numerical simulations of the original passive scalar equation. Effective diffusivity values obtained with our formulas are in good agreement with the measured values. In all cases considered, the error is always smaller than 10% for moderate sweeping. It becomes smaller than 5% for strong sweeping.

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APPENDIX: SELF-CONSISTENCY EQUATION FOR EFFECTIVE DIFFUSIVITIES

We shall give details of the derivation of the *self-consistent* equation (51) for finding the effective diffusivities in the presence of a mean flow. The passive scalar field $\theta(\mathbf{x},t)$ obeys the equation

$$\partial_t \theta(\mathbf{x},t) + (\mathbf{U}+\mathbf{u})\cdot\nabla\theta(\mathbf{x},t) = D_0\Delta\theta(\mathbf{x},t), \tag{A1}$$

where $\mathbf{U}$ is constant and the field $\mathbf{u}$ is incompressible ($\nabla\cdot\mathbf{u}=0$), homogeneous, stationary, and has zero-average value $\langle\mathbf{u}\rangle=0$. The only hypothesis we shall make on the velocity field $\mathbf{u}$ is that it ensures a diffusive transport for sufficiently large times. As shown in Refs. [19, 20], a sufficient condition is that the variance of the vector potential be finite. This is the case, for example, for usual energy spectra having cutoffs both at large and small scale.

Let us now derive the general equation governing the behavior of the mean scalar field $\langle\theta\rangle$. The original equation (A1) is conveniently rewritten in the compact form

$$\mathcal{L}_0\theta = \mathcal{L}_I\theta, \tag{A2}$$

where

$$\mathcal{L}_0 = \partial_t + \mathbf{U}\cdot\nabla - D_0\Delta, \quad \mathcal{L}_I = -\mathbf{u}\cdot\nabla. \tag{A3}$$

The field θ is decomposed into its average and fluctuating parts as

$$\theta = \langle\theta\rangle + \tilde{\theta}. \tag{A4}$$

The following equations immediately follow from Eqs. (A2) and (A4):

$$\mathcal{L}_0\langle\theta\rangle = \langle\mathcal{L}_I\tilde{\theta}\rangle \equiv \Sigma, \tag{A5a}$$

$$(\mathcal{L}_0 - \mathcal{L}_I)\tilde{\theta} = \mathcal{L}_I\langle\theta\rangle - \langle\mathcal{L}_I\tilde{\theta}\rangle. \tag{A5b}$$

Here $\Sigma(\mathbf{x},t)$ plays essentially the same role as the mass operator in quantum field theory. The solution of Eq. (A5b) can be formally expressed in terms of the (unknown) Green's function G as

$$\tilde{\theta}(1) = \int G(1,2)[\mathcal{L}_I(2)\langle\theta(2)\rangle - \Sigma(2)]d2, \tag{A6}$$

where we have used a simplified notation for the space-time variables $(\mathbf{x}_1,t_1)$ and the Green's function G satisfies the usual equation $[\mathcal{L}_0(1)-\mathcal{L}_I(1)]\ G(1,2)=\delta(1-2)$. Operating with $\mathcal{L}_I(1)$ on Eq. (A6) and taking the average, we obtain the equation for Σ,

$$\Sigma(1) = \int \langle\mathcal{L}_I(1)G(1,2)\mathcal{L}_I(2)\rangle\langle\theta(2)\rangle - \langle\mathcal{L}_I G(1,2)\rangle\Sigma(2)d2. \tag{A7}$$

Equation (A7) can be recast in a more convenient form by using the Dyson equation

$$G(1,2)=G_0(1,2)+\int G_0(1,3)\mathcal{L}_I(3)G(3,2)d3, \quad \text{(A8)}$$

where G_0 is the Green's function of $\mathcal{L}_0$ [i.e., $\mathcal{L}_0(1)G_0(1,2)=\delta(1-2)$]. Equation (A8) is an immediate consequence of the definition of the Green's functions G and G_0.

Inserting Eq. (A8) into Eq. (A7), we finally obtain

$$\Sigma(\mathbf{x},t)=\nabla_\alpha\left[\int d\mathbf{R}\int d\tau\, D_{\alpha\beta}(\mathbf{R},\tau)\nabla_\beta\langle\theta(\mathbf{x}-\mathbf{R};t-\tau)\rangle\right]. \quad \text{(A9)}$$

Here we have used the convolution structure of Eq. (A7), which is due to homogeneity and stationarity and the Fourier transform of $D_{\alpha\beta}$ is defined as

$$\hat{D}_{\alpha\beta}(\mathbf{k},\omega)=-\frac{\hat{R}_{\alpha\beta}(\mathbf{k},\omega)}{1+k_\alpha k_\beta\hat{R}_{\alpha\beta}(\mathbf{k},\omega)}, \quad \text{(A10)}$$

with $\hat{R}_{\alpha\beta}$ the Fourier transform of $-u_\alpha G u_\beta$. Equation (A9) gives for the evolution of the mean scalar field

$$\mathcal{L}_0\langle\theta\rangle=\nabla_\alpha\left[\int d\mathbf{R}\int d\tau\, D_{\alpha\beta}(\mathbf{R},\tau)\nabla_\beta\langle\theta(\mathbf{x}-\mathbf{R};t-\tau)\rangle\right]. \quad \text{(A11)}$$

No approximation has been made to derive this equation, which is, however, only formal since $\hat{D}_{\alpha\beta}$ is not known.

We restrict now Eq. (A11) to analyze the transport on scales large compared to those of the turbulent field $\mathbf{u}$. For these modes the $\hat{D}_{\alpha\beta}(\mathbf{k},\omega)$ appearing in Eq. (A11) is essentially equal to its value at the origin $\mathbf{k}=0$, $\omega=0$. It follows that the mean scalar field evolves according to

$$\partial_t\langle\theta\rangle+\mathbf{U}\cdot\boldsymbol{\nabla}\langle\theta\rangle=D_{\alpha\beta}\nabla_\alpha\nabla_\beta\langle\theta\rangle, \quad \text{(A12)}$$

where the effective diffusivity tensor is

$$D_{\alpha\beta}=\delta_{\alpha\beta}D_0+\int d\mathbf{R}\int d\tau\langle u_\alpha(1)G(1,2)u_\beta(2)\rangle. \quad \text{(A13)}$$

The problem now is to determine the Green's function appearing in Eq. (A13). The DIA provides the closed quadratic equation

$$\langle\hat{G}\rangle(\mathbf{k},\omega)=\hat{G}_0(\mathbf{k},\omega)+\hat{G}_0(\mathbf{k},\omega)\langle\widehat{\mathcal{L}_I\langle G\rangle\mathcal{L}_I}\rangle(\mathbf{k},\omega)\langle\hat{G}\rangle(\mathbf{k},\omega), \quad \text{(A14)}$$

with $\langle\widehat{\mathcal{L}_I\langle G\rangle\mathcal{L}_I}\rangle$ the Fourier transform of $\langle\mathcal{L}_I\langle G\rangle\mathcal{L}_I\rangle$. A possible procedure leading to the DIA equation (A14) from the original exact equation (A5) is the following. Adding and subtracting $\mathcal{L}_I\langle G\rangle$ to and from Eq. (A5a) and using the definition of G, we obtain the relation

$$G=\langle G\rangle+\int G[\mathcal{L}_I\langle G\rangle-\langle\mathcal{L}_I\widetilde{G}\rangle]. \quad \text{(A15)}$$

Let us now treat perturbatively the second term on the right-hand side, stop at first order, and use the resulting expression to calculate $\langle\mathcal{L}_I G\rangle$, which appears in the exact Dyson equation

$$\langle G\rangle=G_0+\int G_0\langle\mathcal{L}_I G\rangle. \quad \text{(A16)}$$

The final result expressed in Fourier space is exactly the DIA equation (A14).

The Fourier transform $\hat{C}_{\alpha\beta}(\mathbf{k},\tau)$ of the two point correlation function $\langle u_\alpha(1)u_\beta(2)\rangle$ for the velocities, in an incompressible, homogeneous, and isotropic medium is given by

$$\hat{C}_{\alpha\beta}(\mathbf{k},\tau)=(\delta_{\alpha\beta}k^2-k_\alpha k_\beta)g(k,\tau). \quad \text{(A17)}$$

Let us consider for simplicity a separable expression for $g(k,\tau)$:

$$g(k,\tau)=f(k)e^{-|\tau|/\tau_0},$$

where an exponential behavior for the time dependence of the two-point correlation function is also assumed.

After inserting the above expression for $\hat{C}_{\alpha\beta}(\mathbf{k},\tau)$ into Eq. (A14), we obtain

$$\langle\hat{G}\rangle(\mathbf{k},\omega)=\left[\hat{G}_0^{-1}(\mathbf{k},\omega)+\frac{k_\alpha k_\beta}{(2\pi)^d}\int d\mathbf{q}(q^2\delta_{\alpha\beta}-q_\beta q_\alpha)f(q)\times\langle\hat{G}\rangle(\mathbf{k}-\mathbf{q},\omega+1/\tau_0)\right]^{-1}, \quad \text{(A18)}$$

where d is the space dimension. Such an expression is continued fraction type of equation with positive terms [i.e., $q^{(1+d)}f(q)$ is positive since it is proportional to the energy spectrum of the turbulent fluctuations]. Thus it can easily be solved numerically by the method of ''chain fractions'' [13].

In the infrared limit, it is easy to verify that Eq. (A18) becomes

$$\langle\hat{G}\rangle(\mathbf{k},\omega)=[i\omega+i\mathbf{k}\cdot\mathbf{U}+D_{\alpha\beta}k_\alpha k_\beta]^{-1}, \quad \text{(A19)}$$

where $D_{\alpha\beta}$ is given by Eq. (A13) after substituting G with the Fourier transform of $\langle\hat{G}\rangle(\mathbf{k},1/\tau_0)$. Equation (A13) thus becomes an equation for finding $D_{\alpha\beta}$:

$$D_{\alpha\beta}=\delta_{\alpha\beta}D_0+\frac{1}{(2\pi)^d}\int d\mathbf{q}\,\frac{(q^2\delta_{\alpha\beta}-q_\alpha q_\beta)f(q)}{1/\tau_0+i\mathbf{q}\cdot\mathbf{U}+D_{rs}q_r q_s}. \quad \text{(A20)}$$

Such a method for finding $D_{\alpha\beta}$ is called *self-consistent* [21]. There are two steps to obtain Eq. (A20): First, G has been substituted by $\langle G\rangle$ in the exact equation (A13); second the infrared approximation for $\langle G\rangle$ has been taken. It is quite evident that the latter approximation works better and better as the spectrum of turbulence decreases quickly.

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